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Hyperinvariant subspaces for operators having a compact part

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ABSTRACT

It is well known that if $T = A \oplus B$, where A is compact, then T has a nontrivial hyperinvariant subspace. In this paper, we try to solve the hyperinvariant subspace problem for operators which have a compact part. Our main result is that if A is compact, then either $\begin{pmatrix} A & * \\ 0 & B \end{pmatrix}$ or $\begin{pmatrix} A & 0 \\ * & B \end{pmatrix}$ has a nontrivial hyperinvariant subspace.

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space and $\mathcal{L}(H)$ be the algebra of all bounded linear operators acting on H. The commutant of T, denoted by $\{T\}'$, is the algebra of all operators X in $\mathcal{L}(H)$ such that $XT = TX$. A subspace $M \subset H$ is called a *nontrivial hyperinvariant subspace* for T if $\{0\} \neq M \neq H$ and $XM \subseteq M$ for each $X \in \{T\}'$. In particular, if $TM \subseteq M$, then the subspace M is called a *nontrivial invariant subspace* for T. The *hyperinvariant subspace problem* is the question of whether every operator in $\mathcal{L}(H) \backslash \mathbb{C}$ has a nontrivial hyperinvariant subspace. An operator $T \in \mathcal{L}(H)$ is called a *quasinilpotent operator* if $\sigma(T) = \{0\}$, where $\sigma(\cdot)$ means spectrum. In this case, the sequence $\{\|T^n\|^{\frac{1}{n}}\}$ converges to zero as $n \to \infty$. An operator is called a *quasiaffinity* if it is a one–one mapping having dense range.

Now, let $T \in \mathcal{L}(H)$ be an operator which has a compact part, that is, T is an operator of the form $\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$, where A is a compact operator. It is well known by [3, Theorem 1.4] that if $C = 0$, then T has a nontrivial hyperinvariant subspace. In this paper we examine the following question.

Does T have a nontrivial hyperinvariant subspace if C is nonzero? (1)

In 1972, R.G. Douglas and C. Pearcy [3] showed that the answer of the question (1) is "partially" affirmative.

Theorem 1.1. *(See [3, Theorem 2.7].) Suppose M is a nontrivial subspace of H. Let $A \in \mathcal{L}(M)$ and $B \in \mathcal{L}(M^{\perp})$. If A is compact and there exists a quasiaffinity $J : M^{\perp} \to M$ such that $AJ = JB$, then T has a nontrivial hyperinvariant subspace.*

On the other hand, in [4] we obtained an affirmative answer of the question (1) if the $(1,2)$-entry of T^n is sufficiently "small" in some sense. Write $T^n = \begin{pmatrix} A^n & C_n \\ 0 & B^n \end{pmatrix} \begin{matrix} M \\ M^{\perp} \end{matrix}$. Then we have:

Theorem 1.2. *(See [4, Corollary 2.7].) With the above notation, if $\|C_n^* x\| \leqslant d\|A^{*n} x\|$, where $d > 0$ for all $x \in M$ and all n, then T has a nontrivial hyperinvariant subspace.*

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In this paper, we will show that: if A is compact, then either $\left(\begin{smallmatrix} A & * \\ 0 & B \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} A & 0 \\ * & B \end{smallmatrix}\right)$ has a nontrivial hyperinvariant subspace. And then we provide sufficient conditions that the answer of the above question (1) is affirmative.

2. Operators having a compact part

Assume that T has dense range. Choose a unit vector $x_0 \in H$ and $0 < \varepsilon < 1$. If $\mathcal{F} = \{y \in H\colon\ \|Ty - x_0\| \leqslant \varepsilon\}$, then $\mathcal{F}$ is a nonempty, norm closed and convex set. So there exists the unique minimal vector $y_0 = y_0(x_0, \varepsilon) \in \mathcal{F}$. We say that y_0 is the *extremal (minimal) vector* for T, x_0, ε. In this case, $\|Ty_0 - x_0\| = \varepsilon$. In [1], S. Ansari and P. Enflo established an important equation on extremal vectors called "Orthogonality Equation".

Lemma 2.1 *(Orthogonality equation). If y_0 is the extremal vector for T, x_0, ε, then*

$$T^*(x_0 - Ty_0) = \delta y_0, \quad \delta > 0.$$

Lemma 2.2. *Let y_n be the extremal vector for T^n, x_0, ε. If T is a contraction, then $\|y_n\| \leqslant \|y_{n+1}\|$.*

Proof. Observe that

$$\left\|T^{n+1}y_{n+1} - x_0\right\| = \left\|T^n(Ty_{n+1}) - x_0\right\| = \varepsilon.$$

Since y_n is the minimal vector satisfying $\|T^n y_n - x_0\| = \varepsilon$, we have $\|y_n\| \leqslant \|Ty_{n+1}\| \leqslant \|y_{n+1}\|$. □

Our main result follows,

Theorem 2.3. *Suppose M is a nontrivial subspace of H. Let $A \in \mathcal{L}(M)$ and $B \in \mathcal{L}(M^\perp)$. If A is compact, then either $\left(\begin{smallmatrix} A & * \\ 0 & B \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} A & 0 \\ * & B \end{smallmatrix}\right)$ has a nontrivial hyperinvariant subspace.*

Proof. Let A and B be contractions. Since every eigenspace of operators is clearly a nontrivial hyperinvariant subspace, we can assume that A^* and B^* are injective, and A is a quasinilpotent operator. Choose unit vectors $x_0 \in M$, $x_1 \in M^\perp$, and $0 < \varepsilon < 1$. Since A and B have dense ranges, we can consider extremal vectors of the operators. Let y_n be the extremal vector for A^n, x_0, ε and, similarly, z_n be extremal vectors for B^n, x_1, ε. There are two cases to consider.

(Case 1) There exists a natural number N such that if $n > N$, then $\|z_n\| \geqslant c\|z_{n+1}\|$ for some $c > 0$.

In this case we claim that

$$\lim_{n\to\infty} \frac{\|z_n\|}{\|y_n\|} = 0.$$

Indeed, $\|z_n\| \leqslant c^{n-1}\|z_1\|$ and $1 - \varepsilon \leqslant \|A^n y_n\| \leqslant \|A^n\|\|y_n\|$, so that

$$\frac{\|z_n\|}{\|y_n\|} \leqslant \alpha c^n \|A^n\|, \quad \alpha = \frac{\|z_1\|}{(1-\varepsilon)c}.$$

Since A is a quasinilpotent operator, $\{c^n\|A^n\|\}$ converges to zero as $n \to \infty$. Hence by Lemma 2.2, we have

$$\lim_{n\to\infty} \frac{\|z_n\|}{\|y_{n+1}\|} = 0. \tag{2}$$

Choose a subsequence $\{n_k\}$ such that $\{B^{n_k} z_{n_k}\}$ and $\{A^{n_k+1} y_{n_k+1}\}$ converge weakly to s_0 and t_0, respectively. Then we can easily show that s_0 and t_0 are nonzero. Write $s_k := B^{n_k} z_{n_k} \in M^\perp$, $t_k := A^{n_k+1} y_{n_k+1} \in M$ and $T := \left(\begin{smallmatrix} A & 0 \\ * & B \end{smallmatrix}\right)$. We now claim that

$$\left\langle Xs_k, T^*(x_0 - t_k)\right\rangle \to 0 \quad \text{for each contraction } X \in \{T\}'. \tag{3}$$

Let

$$Xz_{n_k} := \alpha_k y_{n_k+1} + \omega_k, \quad \text{where } \omega_k \perp y_{n_k+1}.$$

Then

$$\|z_{n_k}\|^2 \geqslant |\alpha_k|^2 \|y_{n_k+1}\|^2 + \|\omega_k\|^2,$$

which gives

$$|\alpha_k| \leqslant \frac{\|z_{n_k}\|}{\|y_{n_k+1}\|} \to 0 \tag{4}$$

by (2). On the other hand,

$$\langle Xs_k, T^*(x_0 - t_k)\rangle = \langle \alpha_k y_{n_k+1}, T^{*n_k+1}(x_0 - t_k)\rangle + \langle \omega_k, T^{*n_k+1}(x_0 - t_k)\rangle.$$

By the orthogonality equation in Lemma 2.1, we have $T^{*n_k+1}(x_0 - t_k) = A^{*n_k+1}(x_0 - t_k) = \delta_{n_k+1} y_{n_k+1}$, and hence $\langle \omega_k, T^{*n_k+1}(x_0 - t_k)\rangle = 0$. Therefore

$$\langle Xs_k, T^*(x_0 - t_k)\rangle = \langle \alpha_k y_{n_k+1}, T^{*n_k+1}(x_0 - t_k)\rangle = \alpha_k \langle A^{n_k+1} y_{n_k+1}, x_0 - t_k\rangle.$$

But since $\|A^{n_k+1} y_{n_k+1}\| < 1$ and $\|x_0 - t_k\| = \varepsilon$, it follows from (4) that

$$\left|\langle Xs_k, T^*(x_0 - t_k)\rangle\right| \leqslant \varepsilon|\alpha| \to 0$$

which proves (3). Moreover, since A^* is compact and $\{t_k\}$ converges to t_0 weakly, the sequence $\{T^*(x_0 - t_k)\}$ converges to $T^*(x_0 - t_0)$ in norm. Then by (3) we have

$$\langle Xs_0, T^*(x_0 - t_0)\rangle = 0 \quad \text{for all } X \in \{T\}'.$$

Note that $(x_0 - t_0)$ is a nonzero vector, so is $T^*(x_0 - t_0)$. Indeed, we obtain

$$\varepsilon^2 = \|x_0 - t_k\|^2 = \langle x_0, x_0 - t_k\rangle - \langle t_k, x_0 - t_k\rangle,$$

so that $\langle x_0, x_0 - t_k\rangle = \varepsilon^2 + \delta_{n_k+1}\|y_{n_k+1}\|^2 > 0$ for each k. Also, since s_0 is a nonzero vector, we can say that $N \equiv \mathrm{cl}\{T\}' s_0$ is a nontrivial hyperinvariant subspace for $T = \left(\begin{smallmatrix} A & 0 \\ * & B \end{smallmatrix}\right)$.

(Case 2) There exists a subsequence $\{n_k\}$ such that

$$\frac{\|z_{n_k}\|}{\|z_{n_k+1}\|} \to 0. \tag{5}$$

Firstly we assume that there exists a natural number N such that if $n > N$, then $\|y_n\| \geqslant c\|z_n\|$ for some $c > 0$. Choose a subsequence $\{n_k\}$ such that $\{s_k := B^{n_k} z_{n_k}\}$ and $\{t_k := A^{n_k+1} y_{n_k+1}\}$ converge weakly to s_0 and t_0, respectively and (5) holds. Then we have

$$\frac{\|z_{n_k}\|}{\|y_{n_k+1}\|} \leqslant \frac{\|z_{n_k}\|}{c\|z_{n_k+1}\|} \to 0.$$

Therefore by the same argument as in (Case 1), we get the result. On the other hand, assume that there exists a subsequence $\{n_j\}$ such that

$$\frac{\|y_{n_j+1}\|}{\|z_{n_j+1}\|} \to 0.$$

Choose a subsequence $\{n_k\}$ of $\{n_j\}$ such that $\{s_k := A^{n_k} y_{n_k}\}$ and $\{t_k := B^{n_k+1} z_{n_k+1}\}$ converge weakly to s_0 and t_0, respectively and $\frac{\|y_{n_k+1}\|}{\|z_{n_k+1}\|} \to 0$. Then by Lemma 2.2 we have

$$\frac{\|y_{n_k}\|}{\|z_{n_k+1}\|} \leqslant \frac{\|y_{n_k+1}\|}{\|z_{n_k+1}\|} \to 0.$$

Write $T := \left(\begin{smallmatrix} A & * \\ 0 & B \end{smallmatrix}\right)$. Then the same argument as in (Case 1) we have

$$\langle XTs_k, x_1 - t_k\rangle \to 0 \quad \text{for each contraction } X \in \{T\}'. \tag{6}$$

Since the sequence $\{Ts_k\}$ converges to a nonzero vector Ts_0 in norm and $x_1 - t_0$ is nonzero, it follows from (6) that

$$\langle XTs_0, x_1 - t_0\rangle = 0,$$

and hence $N \equiv \mathrm{cl}\{T\}' Ts_0$ is a nontrivial hyperinvariant subspace for $T = \left(\begin{smallmatrix} A & * \\ 0 & B \end{smallmatrix}\right)$. This complete the proof. □

Let $T \in \mathcal{L}(H)$ be a quasinilpotent operator with dense range and y_n be the extremal vector for T^n, x_0, ε. It was shown in [1, Lemma 1] that for each $x_0 \in H$ and $0 < \varepsilon < 1$ there exists a subsequence $\{y_{n_k}\}$ such that

$$\lim_{k\to\infty} \frac{\|y_{n_k}\|}{\|y_{n_k+1}\|} = 0. \tag{7}$$

However, it can be found in Section 3.2 of [2] that Eq. (7) is not unique property to quasinilpotent operators. For example, define a sequence $\{\alpha_n\}$ by

$$\alpha_n = \begin{cases} \frac{1}{k} & \text{if } n = 2^k \text{ for some } k \in \mathbb{N}, \\ 1 & \text{otherwise.} \end{cases}$$

Let T be the backward weighted shift defined by the equation $Te_n = \alpha_n e_{n-1}$ ($n \geqslant 1$), where $\{e_n\}$ is the orthonormal basis of H. For $x_0 = e_0$ and $0 < \varepsilon < 1$, the extremal vector y_n is written by

$$y_n = \frac{1-\varepsilon}{\alpha_1\alpha_2\cdots\alpha_n} e_n.$$

Thus if $n_k = 2^k - 1$, then

$$\frac{\|y_{n_k}\|}{\|y_{n_k+1}\|} = \alpha_{n_k+1} = \frac{1}{k} \to 0,$$

so that T satisfies (7). However, T is not a quasinilpotent operator. Indeed, observe that the norm of T^n is given by

$$\|T^n\| = \sup_l \left| \prod_{i=1}^n \alpha_{l+i} \right|. \tag{8}$$

Then for fixed n, choose k such that $2^k \geqslant n+1$ and let $l = 2^k$. Since $\alpha_{l+i} = 1$ for all $i = 1, 2, \ldots, n$. It follows from (8) that $\|T^n\| \geqslant 1$. But since $\|T\| = 1$, we have $\|T^n\| = 1$ for each n. Therefore $r(T) = 1$, and hence T is not a quasinilpotent operator.

Definition 2.4. An operator $T \in \mathcal{L}(H)$ with dense range is called a *weakly quasinilpotent operator* if for each unit vector $x_0 \in H$ and $0 < \varepsilon < 1$, there exists a subsequence $\{n_k\}$ such that

$$\lim_{k\to\infty} \frac{\|y_{n_k}\|}{\|y_{n_k+1}\|} = 0,$$

where y_n is the extremal vector for T^n, x_0, ε.

Every quasinilpotent operator is a weakly quasinilpotent operator, but we do not know whether the converse is true or not. But it is easy to find operators which are not weakly quasinilpotent operator, for example, invertible operators or nonzero normal operators.

Now, the following corollaries give partial solutions of the question (1).

Corollary 2.5. *Let $T \in \mathcal{L}(H)$ have a compact part. If T^* is not a weakly quasinilpotent operator, then T has a nontrivial hyperinvariant subspace.*

Proof. Assume that T is an injective contraction. Let T be of the form $\left(\begin{smallmatrix} A & * \\ 0 & * \end{smallmatrix}\right)$, where A is compact. Since every eigenspace of A is clearly a nontrivial hyperinvariant subspace for T, we can assume that A^* is a quasinilpotent operator. Moreover, since T^* has dense range so does A. Choose unit vectors $x_0 \in M$ and $0 < \varepsilon < 1$. Let y_n be the extremal vector for A^{*n}, x_0, ε. Since T^* is not a weakly quasinilpotent operator, there exist a unit vector $x_1 \in H$ and $0 < \varepsilon_1 < 1$ such that z_n be the extremal vector for T^{*n}, x_1, ε_1 satisfying $\|z_n\| \geqslant c\|z_{n+1}\|$ for some $c > 0$. Then by (Case 1) in the proof of Theorem 2.3, we have

$$\lim_{n\to\infty} \frac{\|z_n\|}{\|y_{n+1}\|} = 0.$$

Choose a subsequence $\{n_k\}$ such that $\{T^{*n_k} z_{n_k}\}$ and $\{A^{n_k+1} y_{n_k+1}\}$ converge weakly to s_0 and t_0, respectively. Then by the same argument as in (Case 1) in the proof of Theorem 2.3, $N \equiv \mathrm{cl}\{T^*\}' s_0$ is a nontrivial hyperinvariant subspace for T^*. However, the existence of nontrivial hyperinvariant subspace is invariant for adjoint, so that T has a nontrivial hyperinvariant subspace. □

In Corollary 2.5, if we give the condition "not weakly quasinilpotent" to a part of T^*, the conclusion remains still true.

Corollary 2.6. *Let T be of the form $\left(\begin{smallmatrix} A & * \\ 0 & B \end{smallmatrix}\right)$, where A is compact and B is injective. If B^* is not a weakly quasinilpotent operator, then T has a nontrivial hyperinvariant subspace.*

Proof. Immediate from (Case 1) in the proof of Theorem 2.3. □

We conclude with a nontrivial example revealing Corollary 2.6. An operator T is said to be *hyponormal* if $T^*T - TT^*$ is positive semi-definite.

Example 2.7. Define a bilateral sequence $\{\alpha_n\}$ by

$$\alpha_n := \begin{cases} 2^{n-1} & \text{if } n < 0, \\ 1 - (\frac{1}{2})^{n+1} & \text{if } n \geqslant 0. \end{cases}$$

Let B be the bilateral weighted shift defined by the equation $Be_n = \alpha_{n+1} e_{n+1}$ $(n \in \mathbb{Z})$, where $\{e_n\}$ is the orthonormal basis of $H := \ell^2(\mathbb{Z})$. Then evidently, B is an injective contraction. Moreover, since the weight $\{\alpha_n\}$ of B is increasing, it follows that B is a hyponormal operator. Let y_n be the extremal vector for B^{*n}, e_0, ε. Then y_n is written by

$$y_n = \frac{1-\varepsilon}{\alpha_1 \alpha_2 \cdots \alpha_n} e_n.$$

Thus for each n,

$$\frac{\|y_n\|}{\|y_{n+1}\|} = \alpha_{n+1} \geqslant \frac{1}{2},$$

so that B^* is not a weakly quasinilpotent operator. Therefore if A is compact, then by Corollary 2.6 every operator of this form $\left(\begin{smallmatrix} A & * \\ 0 & B \end{smallmatrix}\right)$ has a nontrivial hyperinvariant subspace.

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